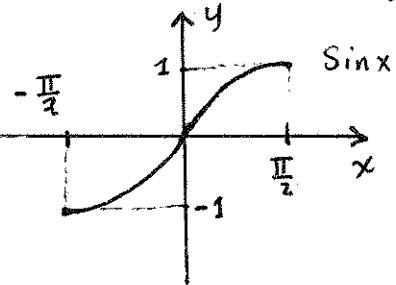


## Section 6.6 Inverse trigonometric Functions

Recall that trig. functions ( $\sin x, \cos x, \dots$ ) are not one-to-one. So, we must restrict their domain in order to find their inverses.

SINE Let  $f(x) = \sin x$ , for  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ .  $f$  is one-to-one on this domain. The inverse function is called the "inverse sine function" or the "arcsine function". It is denoted by  $f^{-1}(x) = \sin^{-1}(x)$  or arcsin( $x$ ).



Definition:  $\sin^{-1}(x) = y \Leftrightarrow \sin(y) = x$

For  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ ,  $-1 \leq \sin x \leq 1$  (Range of Sine is  $[-1, 1]$ )

Thus, domain of  $\sin^{-1}$  is  $[-1, 1]$ .

Examples ①  $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$  Since  $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

②  $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$  Since  $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

Example. Find  $\tan(\sin^{-1}(\frac{1}{3}))$ . Let  $\theta = \sin^{-1}(\frac{1}{3})$ . Then,

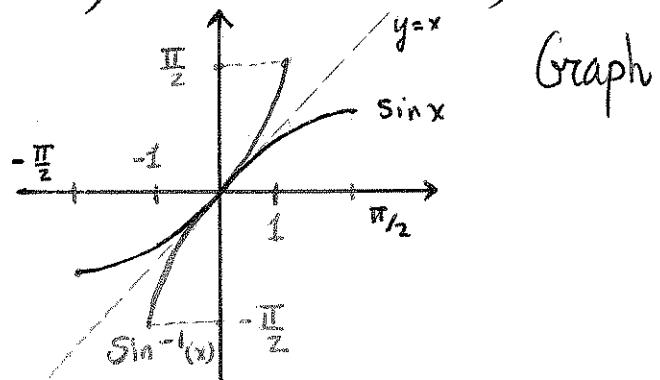
$$\sin \theta = \frac{1}{3}$$

$$\Rightarrow \tan \theta = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$$

Cancellation Equations:  $\sin(\sin^{-1}(x)) = x$ ,  $\sin(\sin^{-1}(x)) = x$

Continuity:  $\sin^{-1}(x)$  is continuous

Since  $\sin(x)$  is continuous.



$$\text{Differentiation: } \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

proof: Let  $y = \sin^{-1} x$ ; then,  $\sin(y) = x \leftarrow \text{take derivative}$

$$\cos(y) \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}};$$

This is because  $\cos(\sin^{-1}(x)) = \sqrt{1-\sin^2(\sin^{-1}(x))} = \sqrt{1-x^2}$  ■

Example: Let  $f(x) = \sin^{-1}(x^2-1)$ . Find  $f'(x)$ , and Domain of  $f$ .

Domain of  $f$ :  $-1 \leq x^2-1 \leq 1 \Rightarrow 0 \leq x^2 \leq 2 \Rightarrow -\sqrt{2} \leq x \leq \sqrt{2}$

$$f'(x) = \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot \frac{d}{dx}(x^2-1) = \frac{2x}{\sqrt{1-(x^2-1)^2}}. \text{ Domain of } f', -\sqrt{2} < x < \sqrt{2}.$$

COSINE:  $\cos^{-1}(x) = y \Leftrightarrow \cos(y) = x$ ; Domain:  $[-1, 1]$ , Range:  $[0, \pi]$

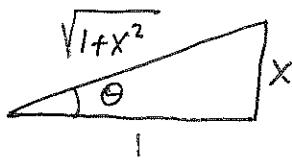
$$\frac{d}{dx} \cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

TANGENT: Let  $f(x) = \tan x$ ,  $-\frac{\pi}{2} < x < \frac{\pi}{2}$ . Then  $f$  is one-to-one

$$f^{-1}(x) = \tan^{-1}(x) \text{ or } \arctan(x).$$

Range of  $\tan(x)$  is  $\mathbb{R} = (-\infty, \infty) \Rightarrow$  Domain of  $\tan^{-1}(x)$  is  $\mathbb{R}$ .

Example: Simplify  $\cos(\tan^{-1}(x))$ . Let  $\theta = \tan^{-1}(x) \Rightarrow \tan \theta = x$

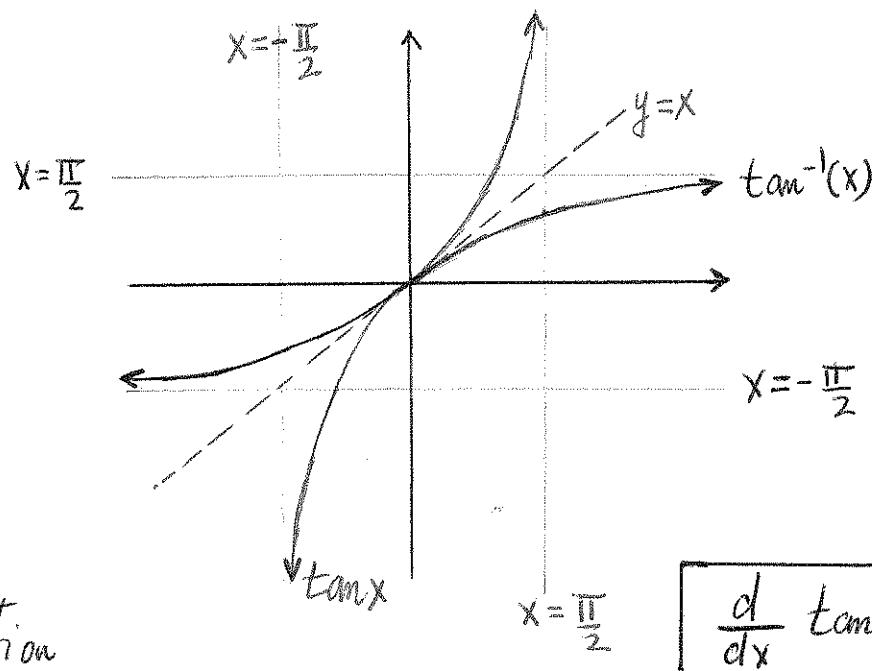


$$\text{Thus } \cos(\theta) = \frac{1}{\sqrt{x^2+1}}$$

Limits:  $\lim_{x \rightarrow -\infty} \tan^{-1}(x) = -\frac{\pi}{2}$  (since  $\lim_{x \rightarrow -\frac{\pi}{2}^+} \tan(x) = -\infty$ )

and  $\lim_{x \rightarrow +\infty} \tan^{-1}(x) = \frac{\pi}{2}$  (since  $\lim_{x \rightarrow \frac{\pi}{2}^-} \tan(x) = +\infty$ )

Graph



Differentiation

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

Let  $y = \tan^{-1}(x) \Rightarrow \tan(y) = x \Rightarrow \sec^2(y) \frac{dy}{dx} = 1$ ; Thus

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{\sec^2(\tan^{-1}(x))} = \frac{1}{1 + \tan^2(\tan^{-1}x)} = \frac{1}{1+x^2}$$

COTANGENT Domain of  $\cot^{-1} = \mathbb{R}$ , Range is  $(0, \pi)$

Derivative  $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$

SECANT Domain of  $\sec^{-1} = (-\infty, -1] \cup [1, \infty)$  ( $|x| \geq 1$ )

Range of  $\sec^{-1} = [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

Examples ①  $\sec^{-1}(2) = \frac{\pi}{3}$  Since  $\cos(\frac{\pi}{3}) = \frac{1}{2}$ .

②  $\sec^{-1}(-\frac{2}{\sqrt{3}}) = \frac{7\pi}{6}$  Since  $\cos(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2}$

Differentiation  $\frac{d}{dx} \sec^{-1} x = \frac{1}{x \sqrt{x^2-1}}$

CSCANT Domain of  $\csc^{-1} = \{|x| > 1\}$ , Range  $= (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

Differentiation  $\frac{d}{dx} \csc^{-1} x = \frac{-1}{x \sqrt{x^2-1}}$